# On the Path Integral Approach to Wigner-Dunkl Quantum Mechanics

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# Abstract

Both, the formal Feynman path-integral approach as well as the properly defined Feynman-Kac formula are discussed within the framework of the Wigner-Dunkl deformed quantum mechanics (WDQM). We first revisit both formulations for the standard Heisenberg-Schrödinger quantum dynamics (HSQM) and then apply them to the WDQM. Whereas in HSQM the Euclidean path integral is based on the Wiener measure associated with the Wiener process of Brownian motion, in WDQM the Feynman-Kac formula is based on the Dunkl process, a càdlàg modification of a Wiener process with jumps. The Dunkl process can be viewed as a combination of two Bessel processes being continuous Markov processes.

# Outline

# Feynman's path integral in HSQM

**Hilbert space:** $\mathcal{H} := L^2(\mathbb{R}^d)$

**Hamiltonian:** $H := -\frac{1}{2}\Delta + V(\mathbf{x})$ with units $\hbar = 1 = m$

**Propagator:** $K(\mathbf{y}, \mathbf{x}, t) := \langle \mathbf{y} | \exp\{-\mathrm{i}Ht\} | \mathbf{x} \rangle$

**Formal path integral:**

$$K(\mathbf{y}, \mathbf{x}, t) = \int_{\mathbf{x}(0)=\mathbf{x}}^{\mathbf{x}(t)=\mathbf{y}} \mathcal{D}[x(s)] \exp\left\{\mathrm{i}\int_0^t \mathrm{d}s \left[\frac{1}{2}\dot{\mathbf{x}}^2(s) - V[\mathbf{x}(s)]\right]\right\}$$

Is ill-defined as $\mathcal{D}[x(s)]$ is NOT a measure.
What is the space of paths?

$\Longrightarrow$ Time slicing approach

# Time slicing approach

Recall: $H = T + V$ with $T := -\frac{1}{2}\Delta$, $V$ self-adjoint

Lie-Trotter: $\mathrm{e}^{-\mathrm{i}Ht} = \lim_{N\to\infty}\left(\mathrm{e}^{-\mathrm{i}T\varepsilon}\mathrm{e}^{-\mathrm{i}V\varepsilon}\right)^N$, $\varepsilon := \frac{t}{N}$

Resolution of unity: $\int \mathrm{d}\mathbf{x}_j\,|\mathbf{x}_j\rangle\langle\mathbf{x}_j| = 1$, $j = 1, 2, \ldots, N-1$

$$K(\mathbf{y},\mathbf{x},t) = \lim_{N\to\infty}\prod_{j=1}^{N-1}\int \mathrm{d}\mathbf{x}_j\,\prod_{j=1}^{N}\langle\mathbf{x}_j|\exp\{-\mathrm{i}\,T\varepsilon\}|\mathbf{x}_{j-1}\rangle\,\mathrm{e}^{-\mathrm{i}V(\mathbf{x}_{j-1})\varepsilon}$$

$$\langle\mathbf{x}_j|\mathrm{e}^{-\mathrm{i}T\varepsilon}|\mathbf{x}_{j-1}\rangle = (2\pi\mathrm{i}\varepsilon)^{-d/2}\exp\left\{\mathrm{i}\frac{(\mathbf{x}_j-\mathbf{x}_{j-1})^2}{2\varepsilon}\right\},\ \mathbf{x}_0 = \mathbf{x},\ \mathbf{x}_N = \mathbf{y}$$

Fresnel-type integrals, limit may still not exist
Direct integration only for free particle and harmonic oscillator
More solutions via local space-time trans. in 1980s - 1990s

# Euclidean propagator

**Euclidean time:** $\mathrm{i}t \to \tau$

**Euclidean action:** $S_E[\mathbf{x}(\sigma)] := \int_0^\tau \mathrm{d}\sigma \left[\frac{1}{2}\dot{\mathbf{x}}^2(\sigma) + V[\mathbf{x}(\sigma)]\right]$

**Euclidean propagator:** $K_E(\mathbf{y},\mathbf{x},\tau) := \langle \mathbf{y}| \exp\{-H\tau\}|\mathbf{x}\rangle$

$$K_E(\mathbf{y},\mathbf{x},\tau) = \int_{\mathbf{x}(0)=\mathbf{x}}^{\mathbf{x}(\tau)=\mathbf{y}} \mathcal{D}[x(\sigma)] \exp\{-S_E[x(\sigma)]\}$$

**Feynman-Kac formula:**

$$K_E(\mathbf{y},\mathbf{x},\tau) = \int_{\mathcal{C}(\mathbb{R}^d,\mathbf{x})} \mathrm{d}W[\mathbf{x}(\sigma)] \exp\left\{-\int_0^\tau \mathrm{d}\sigma\, V[\mathbf{x}(\sigma)]\right\} \delta(\mathbf{x}(\tau)-y)$$

$W[\mathbf{x}(\sigma)]$: Well-defined Wiener measure on
$\mathcal{C}(\mathbb{R}^d,\mathbf{x})$: Space of paths in $\mathbb{R}^d$ starting at $\mathbf{x}$

# Wiener process

**Generator:** $\mathcal{L}_W := \frac{1}{2}\Delta$

**Transition density:** $w_\tau^{(d)}(\mathbf{y},\mathbf{x}) := (2\pi\tau)^{-d/2}\exp\{-(\mathbf{y}-\mathbf{x})^2/2\tau\}$

**Interrelation:**

$$\langle \mathbf{y}|\exp\{\mathcal{L}_W\tau\}|\mathbf{x}\rangle = w_\tau^{(d)}(\mathbf{y},\mathbf{x}) = \int_{\mathcal{C}(\mathbb{R}^d,\mathbf{x})} \mathrm{d}W[\mathbf{x}(\sigma)]\delta(\mathbf{x}(\tau)-y)$$

**Stationary Markov process:**

- Positivity: $w_\tau^{(d)}(\mathbf{y},\mathbf{x}) > 0$
- Initial condition: $\lim_{\tau\searrow 0} w_\tau^{(d)}(\mathbf{y},\mathbf{x}) = \delta(\mathbf{y}-\mathbf{x})$
- Normalisation: $\int_{\mathbb{R}^d} \mathrm{d}\mathbf{y}\, w_\tau^{(d)}(\mathbf{y},\mathbf{x}) = 1$
- Convolution: $\int_{\mathbb{R}^d} \mathrm{d}\mathbf{y}\, w_{\tau_2}^{(d)}(\mathbf{z},\mathbf{y}) w_{\tau_1}^{(d)}(\mathbf{y},\mathbf{x}) = w_{\tau_1+\tau_2}^{(d)}(\mathbf{z},\mathbf{x})$

# Bessel process on positive half-line

**Generator:** $\mathcal{L}_{B(\alpha)} := \frac{1}{2}\left(\partial_x^2 + \frac{2\alpha+1}{x}\partial_x\right)$, Index $\alpha > -1$, $x \in \mathbb{R}^+$

**Transition density:** $b_\tau^{(\alpha)}(y,x) := \frac{1}{2\tau}(xy)^{-\alpha}\mathrm{e}^{-(y^2+x^2)/2\tau}\mathrm{I}_\alpha\left(\frac{xy}{\tau}\right)$

**Speed measure:** $\mathrm{d}\mu_{B(\alpha)}(x) := 2x^{2\alpha+1}\mathrm{d}x$

**Convolution:**

$$\int_0^\infty \mathrm{d}\mu_{B(\alpha)}(y)\, b_{\tau_2}^{(\alpha)}(z,y) b_{\tau_1}^{(\alpha)}(y,x) = b_{\tau_1+\tau_2}^{(\alpha)}(z,x)$$

**Relation to Wiener process in $\mathbb{R}^d$:** $x = |\mathbf{x}|$, $y = |\mathbf{y}|$

$$b_\tau^{(\frac{d}{2}-1)}(y,x) = \int_{S^{d-1}} \mathrm{d}\Omega(\mathbf{y})\, w_\tau^{(d)}(\mathbf{y},\mathbf{x})$$

Bessel process for $\alpha = \frac{d}{2} - 1$ is radial part of Wiener process!

# Properties of Bessel processes

**Feynman-Kac:**

$$\langle y|\mathrm{e}^{[\mathcal{L}_{B(\alpha)}-V]\tau}|x\rangle = \int_{\mathcal{C}(\mathbb{R}^+,x)} \mathrm{d}B^{(\alpha)}[x(\sigma)] \exp\left\{-\int_0^\tau \mathrm{d}\sigma\, V[x(\sigma)]\right\} \delta(x(\tau)-y)$$

**Change of index (Radon-Nikodym derivative):**

$$\int_{\mathcal{C}(\mathbb{R}^+,x)} \mathrm{d}B^{(\alpha)}[x(\sigma)] \exp\left\{-\int_0^\tau \mathrm{d}\sigma\, V[x(\sigma)]\right\} \delta(x(\tau)-y) = (xy)^{\beta-\alpha} \times$$
$$\times \int_{\mathcal{C}(\mathbb{R}^+,x)} \mathrm{d}B^{(\beta)}[x(\sigma)] \exp\left\{-\int_0^\tau \mathrm{d}\sigma\, \left[V[x(\sigma)] + \frac{\alpha^2-\beta^2}{2x^2(\sigma)}\right]\right\} \delta(x(\tau)-y)$$

**Special case:** $\alpha = \nu - \frac{1}{2}, \qquad \beta = \nu + \frac{1}{2}$

$$\langle y| \exp\left\{\left(\mathcal{L}_{B^{(\nu-\frac{1}{2})}} - \frac{\nu}{x^2}\right)\tau\right\}|x\rangle = xy\,\langle y|\exp\left\{\mathcal{L}_{B^{(\nu+\frac{1}{2})}}\tau\right\}|x\rangle$$

Explicit integration for radial harmonic oscillator.
Solution for Morse and Coulomb problem via space-time trans.

# Wigner-Dunkl quantum mechanics on real line

**Weighted Hilbert space:** $\mathcal{H} := L^2(\mathbb{R}, |x|^{2\nu}\mathrm{d}x)$

**Dunkl derivative:** $D_x := \partial_x + \frac{\nu}{x}(1-R)$ with $[D_x, x] = 1 + 2\nu R$, Dunkl parameter $\nu > -\frac{1}{2}$ and reflection op. $R$, $Rf(x) := f(-x)$

**Hamiltonian:** $H := -\frac{1}{2}\,D_x^2 + V(x)$

**Free Dunkl particle:** $-\frac{1}{2}D_x^2\,\psi_k(x) = \frac{k^2}{2}\,\psi_k(x), \qquad k \in \mathbb{R}$

with $\quad \psi_k(x) = \frac{|k|^\nu}{c_\nu}E_\nu(\mathrm{i}kx), \qquad c_\nu := \frac{\sqrt{2\pi}}{2^\nu}\frac{\Gamma(2\nu+1)}{\Gamma(\nu+1)}$

**Deformed exponential:** $D_x\,E_\nu(ax) = a\,E_\nu(x)$, $[n]_\nu := n + \nu[1-(-1)^n]$

$$E_\nu(z) := \sum_{n=0}^{\infty}\frac{z^n}{[n]_\nu!} = {}_0F_1(\nu+\tfrac{1}{2},\tfrac{z^2}{4}) + \frac{z}{2\nu+1}\,{}_0F_1(\nu+\tfrac{3}{2},\tfrac{z^2}{4})\,, z \in \mathbb{C}.$$

$$E_\nu(x) = \frac{\Gamma(\nu+\frac{1}{2})}{(|x|/2)^{\nu-\frac{1}{2}}}\left[\mathrm{I}_{\nu-\frac{1}{2}}(|x|) + \mathrm{sgn}\,(x)\mathrm{I}_{\nu+\frac{1}{2}}(|x|)\right]\,, x \in \mathbb{R}.$$

# Time slicing approach

**Propagator:**

$$K(y,x,t) := \langle y|\exp\{-\mathrm{i}Ht\}|x\rangle = \lim_{N\to\infty}\langle y|\left(\mathrm{e}^{\frac{\mathrm{i}}{2}D_x^2\varepsilon}\mathrm{e}^{-\mathrm{i}V\varepsilon}\right)^N|x\rangle$$

**Completeness:** $\displaystyle\int_{\mathbb{R}} \mathrm{d}x_j\,|x_j|^{2\nu}|x_j\rangle\langle x_j| = 1\,, \qquad j = 1,2,\ldots,N-1$

**Feynman PI:**

$$K(y,x,t) = \lim_{N\to\infty}\prod_{j=1}^{N-1}\int_{\mathbb{R}}\mathrm{d}x_j\,|x_j|^{2\nu}\prod_{j=1}^{N}\langle x_j|\mathrm{e}^{\frac{\mathrm{i}}{2}D_x^2\varepsilon}|x_{j-1}\rangle\mathrm{e}^{-\mathrm{i}V(x_{j-1})\varepsilon}$$

**Free WD propagator:**

$$\begin{aligned}\langle y|\mathrm{e}^{\frac{\mathrm{i}}{2}D_x^2 t}|x\rangle &= \frac{(2\pi)^{\nu+\frac{1}{2}}}{c_\nu}\left(\frac{1}{2\pi\mathrm{i}t}\right)^{\nu+\frac{1}{2}}\mathrm{e}^{\frac{\mathrm{i}}{2t}(y^2+x^2)}E_\nu\left(\frac{xy}{\mathrm{i}t}\right)\\ &= \frac{(2\pi)^{\nu+\frac{1}{2}}}{c_\nu}\left(\frac{1}{2\pi\mathrm{i}t}\right)^{\nu+\frac{1}{2}}\mathrm{e}^{\frac{\mathrm{i}}{2t}(y-x)^2}\,{}_1F_1\left(\nu,2\nu+1,\frac{2\mathrm{i}xy}{t}\right)\end{aligned}$$

Same issues: Integrals and limit may not exist,
direct integration only for free particle and harmonic oscillator

# Harmonic oscillator $V(x) = \frac{1}{2}\omega^2 x^2$

Let $V_j = \frac{1}{4}\omega^2(x_j^2 + x_{j-1}^2)$

Short-time propagator:

$$\langle x_j|\mathrm{e}^{\frac{\mathrm{i}}{2}D_x^2\varepsilon}|x_{j-1}\rangle\mathrm{e}^{-\mathrm{i}V_j\varepsilon} = \frac{1}{c_\nu(\mathrm{i}\varepsilon)^{\nu+\frac{1}{2}}}\mathrm{e}^{\frac{\mathrm{i}}{2\varepsilon}(x_j^2+x_{j-1}^2)(1-\frac{\omega^2\varepsilon^2}{2})}E_\nu\left(\frac{x_j x_{j-1}}{\mathrm{i}\varepsilon}\right)$$

$$\text{with} \quad \varepsilon = \frac{\sin(\omega\varepsilon)}{\omega}\left[1 + O(\varepsilon^2)\right], \quad 1 - \frac{\omega^2\varepsilon^2}{2} = \cos(\omega\varepsilon)\left[1 + O(\varepsilon^2)\right]$$

$$\approx \frac{1}{c_\nu}\left(\frac{\omega}{\mathrm{i}\sin(\omega\varepsilon)}\right)^{\nu+\frac{1}{2}}\mathrm{e}^{\frac{\mathrm{i}\omega}{2}(x_j^2+x_{j-1}^2)\cot(\omega\varepsilon)}E_\nu\left(\frac{\omega x_j x_{j-1}}{\mathrm{i}\sin(\omega\varepsilon)}\right)$$

$$= \langle x_j|\mathrm{e}^{\frac{\mathrm{i}}{2}D_x^2\varepsilon-\frac{\mathrm{i}}{2}\omega^2x^2\varepsilon}|x_{j-1}\rangle$$

For last step see P. Sedaghatnia et al, Ann. Phys. 458 (2023) 169445

Path integration becomes trivial!

# Dunkl process on the real line

**Generator:** $\mathcal{L}_{D^{(\nu)}} := \frac{1}{2}D_x^2 = \frac{1}{2}\left(\partial_x^2 + \frac{2\nu}{x}\partial_x - \frac{\nu}{x^2}(1-R)\right)$, $x \in \mathbb{R}$

**Transition density:** $d_\tau^{(\nu)}(y,x) := \langle y|\mathrm{e}^{\tau\mathcal{L}_{D^{(\nu)}}}|x\rangle = \frac{\mathrm{e}^{-(y^2+x^2)/2\tau}}{c_\nu \tau^{\nu+\frac{1}{2}}}\mathrm{E}_\nu\left(\frac{xy}{\tau}\right)$

**Speed measure:** $\mathrm{d}\mu_{D^{(\nu)}}(x) := |x|^{2\nu}\mathrm{d}x$

**Convolution:** $\int_{-\infty}^{\infty}\mathrm{d}\mu_{D^{(\nu)}}(y)\, d_{\tau_2}^{(\nu)}(z,y)d_{\tau_1}^{(\nu)}(y,x) = d_{\tau_1+\tau_2}^{(\nu)}(z,x)$

Dunkl process is a stationary Markov process on real line. A càdlàg (continue à droite, limite à gauche) generalisation of the Wiener process.

**Relation to Bessel process on half line $\mathbb{R}^+$:**

$$d_\tau^{(\nu)}(y,x) = b_\tau^{(\nu-\frac{1}{2})}(|y|,|x|) + xy\, b_\tau^{(\nu+\frac{1}{2})}(|y|,|x|)$$

Note: $\mathrm{d}\mu_{B^{(\nu-\frac{1}{2})}}(x) := 2x^{2\nu}\mathrm{d}x$ and $\mathrm{d}\mu_{B^{(\nu+\frac{1}{2})}}(x) := 2x^{2\nu+2}\mathrm{d}x$ on $\mathbb{R}^+$

# Relation to Bessel processes

**Projector:** $P_\pm := \frac{1}{2}(1 \pm R)\,, \qquad P_+ + P_- = 1\,, \qquad P_+ P_- = 0$

**Hilbert space:** $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-\,,$

where $\mathcal{H}_\pm := P_\pm \mathcal{H} P_\pm = L^2(\mathbb{R}, f(-x) = \pm f(x))$

**Generator decomp.:** $\mathcal{L}^\pm_{D^{(\nu)}} := P_\pm \mathcal{L}_{D^{(\nu)}} P_\pm$

$$\mathcal{L}^+_{D^{(\nu)}} = \frac{1}{2}\left(\partial_x^2 + \frac{2\nu}{x}\partial_x\right)\,, \qquad \mathcal{L}^-_{D^{(\nu)}} = \frac{1}{2}\left(\partial_x^2 + \frac{2\nu}{x}\partial_x\right) - \frac{\nu}{x^2}$$

**Observation:** $\mathcal{L}^\pm_{D^{(\nu)}} = \mathcal{L}_{B^{(\nu\pm\frac{1}{2})}}$ restricted to $L^2(\mathbb{R}^+)$

For even potentials $V(-x) = V(x)$

$$\begin{aligned}\langle y|\mathrm{e}^{\tau(\mathcal{L}_{D^{(\nu)}}-V)}|x\rangle &= \langle |y| \left| \mathrm{e}^{\tau(\mathcal{L}^+_{D^{(\nu)}}-V)} \right| |x|\rangle + \mathrm{sgn}\,(xy) \langle |y| \left| \mathrm{e}^{\tau(\mathcal{L}^-_{D^{(\nu)}}-V)} \right| |x|\rangle \\ &= \langle |y| \left| \mathrm{e}^{\tau(\mathcal{L}_{B^{(\nu-\frac{1}{2})}}-V)} \right| |x|\rangle + xy \langle |y| \left| \mathrm{e}^{\tau(\mathcal{L}_{B^{(\nu+\frac{1}{2})}}-V)} \right| |x|\rangle\end{aligned}$$

# Feynman-Kac-Dunkl formula

**Feynman-Kac-Dunkl formula:**

$$
\begin{aligned}
&\langle y|\mathrm{e}^{\tau(\frac{1}{2}D_x^2-V)}|x\rangle \\
&\quad= \int_{\mathcal{C}(\mathbb{R},x)} \mathrm{d}D^{(\nu)}[x(\sigma)]\exp\left\{-\int_0^\tau \mathrm{d}\sigma\, V[x(\sigma)]\right\}\delta(x(\tau)-y) \\
&\quad= \int_{\mathcal{C}(\mathbb{R}^+,|x|)} \mathrm{d}B^{(\nu-\frac{1}{2})}[x(\sigma)]\exp\left\{-\int_0^\tau \mathrm{d}\sigma\, V[x(\sigma)]\right\}\delta(x(\tau)-|y|) \\
&\qquad+xy\int_{\mathcal{C}(\mathbb{R}^+,|x|)} \mathrm{d}B^{(\nu+\frac{1}{2})}[x(\sigma)]\exp\left\{-\int_0^\tau \mathrm{d}\sigma\, V[x(\sigma)]\right\}\delta(x(\tau)-|y|)
\end{aligned}
$$

Special case $\nu = 0$ reduces to Wiener process on $\mathbb{R}$.

Explicit integration for harmonic oscillator and inverse square potential based on known results for Bessel processes.

# Harmonic oscillator $V(x) = \frac{1}{2}\omega^2 x^2$

Bessel FK: (see e.g. books by Revuz/Yor or Itô/McKean)

$$\int_{\mathcal{C}(\mathbb{R}^+,x)} \mathrm{d}B^{(\alpha)}[x(\sigma)] \exp\left\{-\int_0^\tau \mathrm{d}\sigma\, \frac{\omega^2}{2} x^2(\sigma)\right\} \delta(x(\tau)-y)$$
$$= \frac{1}{2(xy)^\alpha} \frac{\omega}{\sinh(\omega\tau)} \exp\left\{-\frac{\omega}{2}(x^2+y^2)\coth(\omega\tau)\right\} \mathrm{I}_\alpha\left(\frac{\omega xy}{\sinh(\omega\tau)}\right)$$

Dunkl FK:

$$\int_{\mathcal{C}(\mathbb{R},x)} \mathrm{d}D^{(\nu)}[x(\sigma)] \exp\left\{-\int_0^\tau \mathrm{d}\sigma\, \frac{\omega^2}{2} x^2(\sigma)\right\} \delta(x(\tau)-y)$$
$$= \frac{1}{c_\nu}\left(\frac{\omega}{\sinh(\omega\tau)}\right)^{\nu+\frac{1}{2}} \exp\left\{-\frac{\omega}{2}(x^2+y^2)\coth(\omega\tau)\right\} E_\nu\left(\frac{\omega xy}{\sinh(\omega\tau)}\right)$$

Euclidean version of result on slide 12!

# Inverse square potential $V(x) = \frac{\mu^2}{2x^2}$

Bessel FK: (see change of index on slide 9 with $\beta := \sqrt{\alpha^2 + \mu^2}$)

$$\int_{\mathcal{C}(\mathbb{R}^+,x)} \mathrm{d}B^{(\alpha)}[x(\sigma)] \exp\left\{-\int_0^\tau \mathrm{d}\sigma\, \frac{\mu^2}{2x^2(\sigma)}\right\} \delta(x(\tau)-y)$$
$$= \frac{1}{2\tau(xy)^\beta} \exp\left\{-\tfrac{1}{2\tau}(x^2+y^2)\right\} \mathrm{I}_\beta\left(\tfrac{xy}{\tau}\right)$$

Dunkl FK:

$$\int_{\mathcal{C}(\mathbb{R},x)} \mathrm{d}D^{(\nu)}[x(\sigma)] \exp\left\{-\int_0^\tau \mathrm{d}\sigma\, \frac{\mu^2}{2x^2(\sigma)}\right\} \delta(x(\tau)-y)$$
$$= \frac{\exp\left\{-\frac{1}{2\tau}(x^2+y^2)\right\}}{2\tau} \left[\frac{\mathrm{I}_{\beta-}\left(\frac{|xy|}{\tau}\right)}{|xy|^{\beta_-}} + xy\frac{\mathrm{I}_{\beta^+}\left(\frac{|xy|}{\tau}\right)}{|xy|^{\beta_+}}\right]$$

where $\beta^\pm := \sqrt{(\nu \pm \frac{1}{2})^2 + \mu^2}$

# Summary and Outlook

- The Dunkl process replaces the Wiener process when deforming HSQM to WDQM.
- Dunkl processes are known since (at least) 1998;
  see, e.g., M. Rösler, Comm. Math. Phys. 192 (1998) 519-542.
- Current talk based on J. Phys. A 57 (2024) 075201.
- Close relation between Dunkl and Bessel processes may allow for more explicit solutions.
- For example, $V(x) = \frac{1}{|x|}$ can be path-integrated utilising results of Fischer et al, J. Math. Phys. 36 (1995) 2313-2323.
- Generalisation to $D_x = \partial_x + \frac{\alpha}{x} + \frac{\beta}{x}R - \gamma R\partial_x$ possible ?

# Thanks

**THANKS!**